Comment on `Dirac theory in spacetime algebra'

2002 J. Phys. A: Math. Gen. 35 4791

(http://iopscience.iop.org/0305-4470/35/22/401)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4791–4796 PII: S0305-4470(02)24453-4

COMMENT

Comment on 'Dirac theory in spacetime algebra'

William E Baylis

Physics Department, University of Windsor, Windsor, ON, Canada N9B 3P4

Received 30 April 2001, in final form 8 February 2002
Published 24 May 2002
Online at stacks.iop.org/JPhysA/35/4791

Abstract

In contrast to formulations of the Dirac theory by Hestenes and by the present author, the formulation recently presented by Joyce (Joyce W P 2001 *J. Phys. A: Math. Gen.* **34** 1991–2005) is equivalent to the usual Dirac equation only in the case of vanishing mass. For nonzero mass, solutions to Joyce's equation can be solutions either of the Dirac equation in the Hestenes form or of the same equation with the sign of the mass reversed, and in general they are mixtures of the two possibilities. Because of this relationship, Joyce obtains twice as many linearly independent plane-wave solutions for a given momentum eigenvalue as exist in the conventional theory. A misconception about the symmetry of the Hestenes equation and the geometric significance of the algebraic spinors is also briefly discussed.

PACS numbers: 03.50.Dy, 03.30.+p

The recent formulation by Joyce [1] of what he calls the 'bivector Dirac equation' in the complex Dirac algebra (Clifford's geometric algebra of Minkowski spacetime taken over the complex field) can be decomposed into the Dirac equation in the Hestenes form [2–5] for mass $+m$ plus another equation for mass $-m$. The purpose of this comment is to prove this claim and to discuss a few other aspects of the paper. The approach exploits the power of projectors in geometric algebras. To start, we first review the Dirac equation for a free electron and its formulation in algebraic form.

1. Dirac equation

Joyce's formulation is made in the framework of the complex Dirac algebra, that is, Clifford's geometric algebra [6, 7] of Minkowski spacetime taken over the complex field. A matrix form of this algebra is traditionally used for the Dirac equation. Since the objective is to find a new formulation of the Dirac equation, it is reasonable first to put the traditional formulation directly in algebraic form. For simplicity, we consider only the case of a free electron; the procedure for adding a gauge potential is straightforward and can be undertaken later. The Dirac equation for a free electron in units with $\hbar = c = 1$ can be written as a matrix equation

$$\mathrm{i}\nabla\!\!\!/\,\psi = m\psi \tag{1}$$

where ψ is a four-component complex spinor, and the gradient operator in Minkowski spacetime (sum repeated indices $\mu = 0, 1, 2, 3$)

$$\nabla = \gamma_\mu \partial^\mu \tag{2}$$

(which Joyce denotes by $\boldsymbol{d}$)[1] is an expansion in the 4×4 gamma matrices γ_μ with operator coefficients $\partial^\mu = \partial/\partial x_\mu = \eta^{\mu\nu}\partial/\partial x^\nu = \eta^{\mu\nu}\partial_\nu$. Indices are raised and lowered by the metric tensors $(\eta^{\mu\nu})$ and $(\eta_{\mu\nu})$ of Minkowski spacetime. The algebra of the gamma matrices is that of the vector basis of Clifford's geometric algebra of spacetime, namely

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}. \tag{3}$$

In the algebra, the orthonormal basis vectors of spacetime are therefore taken to be $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, and the explicit matrices that represent them are not important. Joyce has considered two possible choices for the metric, but to avoid unnecessary complications, I take $(\eta^{\mu\nu})$ to be the diagonal matrix

$$(\eta^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1) = (\eta_{\mu\nu}) \tag{4}$$

corresponding to the choice of Joyce's parameter $\eta = -1$.

The geometric algebra generated by associative products of basis vectors $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ that satisfy (3) with (4) is denoted [7] $C\ell_{1,3}$. Four spinors ψ, each satisfying the Dirac equation (1), can be used to construct a 4×4 solution matrix Ψ. Let each column of Ψ be a column spinor ψ that satisfies (1). Then Ψ is the matrix representation of an element of the Dirac algebra $C\ell_{1,3}$ over the complex field $\mathbb{C}$. We construct explicit plane-wave solutions below, but the point made here is that the Dirac equation (1) can be written directly as an *algebraic equation* in $C\ell_{1,3} \otimes \mathbb{C}$

$$\mathrm{i}\nabla\Psi = m\Psi, \quad \Psi \in C\ell_{1,3} \otimes \mathbb{C} \tag{5}$$

that gives up to four independent solutions of the traditional column-spinor equation. This equation remains valid under multiplication from the right by any constant algebraic element. In particular, one can multiply by a projector P, that is a Hermitian idempotent

$$\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^\dagger. \tag{6}$$

For example, columns of a 4×4 matrix representation of the algebraic spinor Ψ can be extracted by the application of projectors $\mathsf{P}(\alpha)$, $\alpha = 0, 1, 2, 3$ defined with diagonal matrix representations $\mathsf{P}(\alpha) = (\delta_{\mu\alpha}\delta_{\nu\alpha})$. Thus, the algebraic equation (5) is equivalent to four copies of the usual spinor equation (1).

The projectors $\mathsf{P}(\alpha)$ depend on the matrix representation. However, as we see below, useful projectors can also be defined algebraically, independent of the representation.

2. Joyce equation

Joyce has formulated a *different* equation in the complex Dirac algebra $C\ell_{1,3} \otimes \mathbb{C}$. His equation (19) for a free electron is

$$\mathrm{i}\nabla\Psi_\mathrm{J} = m\Psi_\mathrm{J}\gamma_0 \qquad \Psi_\mathrm{J} \in C\ell_{1,3}^{+} \otimes \mathbb{C} \tag{7}$$

although Joyce uses the notation e_μ in place of γ_μ. It differs from the Dirac equation (5) by the extra factor of γ_0 on the right. The equations are the same only if $m = 0$. The Joyce equation (7)

[1] Joyce has introduced new notation for a number of terms and has redefined other symbols already in common use. A change that is particularly liable to cause confusion is his redefinition of the wedge and dot products. This comment employs the more established notation of Hestenes, but I shall relate this to the notation of Joyce where appropriate.

is invariant under right multiplication of Ψ_J by any constant element that commutes with γ_0. The spinor can be decomposed into pieces $\Psi_J = \Psi_J \mathsf{P}_{+0} + \Psi_J \mathsf{P}_{-0}$ where the constant elements

$$\mathsf{P}_{\pm 0} = \tfrac{1}{2}(1 \pm \gamma_0) \tag{8}$$

are projectors. Equation (7) then splits into two parts, one equivalent to the usual Dirac equation (1) in algebraic form, and the other similar but with the sign of the mass changed:

$$\mathrm{i}\nabla \Psi_J \mathsf{P}_{\pm 0} = \pm m \Psi_J \mathsf{P}_{\pm 0} \tag{9}$$

where we noted the property that $\gamma_0 \mathsf{P}_{\pm 0} = \pm \mathsf{P}_{\pm 0}$. Thus, the part $\Psi_J \mathsf{P}_{+0}$ is a solution to the Dirac equation (5), but $\Psi_J \mathsf{P}_{-0}$ satisfies an equation with the opposite mass $-m$. However as indicated in equation (7), Joyce has imposed an additional condition on his solutions Ψ_J, namely that they be elements of the *even* subalgebra $C\ell^+_{1,3} \otimes \mathbb{C}$ of the complex Dirac algebra (he refers to Ψ_J as a 'generalized bivector', although it can also contain scalars and four-volume elements). Thus Ψ_J can be expanded

$$\Psi_J = \Psi_J^K \, \Gamma_K^+, \qquad K = 1, 2, \ldots, 8 \tag{10}$$

in the basis $\{\Gamma_K^+\} = \{1, \gamma_\mu \gamma_\nu, \gamma_0\gamma_1\gamma_2\gamma_3\}$ of $C\ell^+_{1,3}$ with $\mu, \nu = 0, 1, 2, 3$ and $\mu < \nu$, where Ψ_J^K are complex scalar functions. However, the part $\Psi_J \mathsf{P}_{+0}$ of the Joyce spinor Ψ_J that is a solution of the Dirac equation (5) cannot be even since if Ψ_J is even, then $\Psi_J \gamma_0$, which is part of $\Psi_J \mathsf{P}_{+0}$, is odd. Indeed, the even and odd parts of $\Psi_J \mathsf{P}_{+0}$ are of the same size. Consequently, there is *no* solution $\Psi_J \in C\ell^+_{1,3} \otimes \mathbb{C}$ that is also a solution of the algebraic Dirac equation (5) if $m > 0$. Nevertheless, this result does not preclude the possibility that there is some other relation between Ψ_J and ψ that would be consistent with both equations (1) and (7). Joyce does not specify an explicit relation between his Ψ_J and the usual Dirac spinor, but we can find one by comparing his equation (7) with that of the Hestenes form.

3. Hestenes equation

Hestenes [2, 5] constructed a form of the Dirac equation in the *real* algebra $C\ell_{1,3}$, which he calls the spacetime algebra (STA). The Hestenes equation for the free electron in the STA is

$$-\nabla \Psi_H \gamma_1 \gamma_2 = m \Psi_H \gamma_0, \qquad \Psi_H \in C\ell^+_{1,3}. \tag{11}$$

The Joyce and Hestenes equations act in different spaces and are not equivalent. To compare them, we consider both acting in the larger space of the complex Dirac algebra $C\ell_{1,3} \otimes \mathbb{C}$.

The Joyce and Hestenes spinors in $C\ell_{1,3} \otimes \mathbb{C}$ are related by the simple projectors

$$\mathsf{P}_{\pm 12} = \tfrac{1}{2}(1 \pm \mathrm{i}\gamma_1\gamma_2) \tag{12}$$

which can be applied to the Joyce equation (7) to give

$$\mathrm{i}\nabla(\Psi_J \mathsf{P}_{\pm 12}) = \mp \nabla(\Psi_J \mathsf{P}_{\pm 12})\gamma_1\gamma_2 = m(\Psi_J \mathsf{P}_{\pm 12})\gamma_0 \tag{13}$$

where we noted

$$\mathsf{P}_{\pm 12} = \pm \mathrm{i} \mathsf{P}_{\pm 12} \gamma_1 \gamma_2. \tag{14}$$

Thus, any spinor solution Ψ_J can be split into parts

$$\Psi_J = \Psi_J \mathsf{P}_{+12} + \Psi_J \mathsf{P}_{-12} \tag{15}$$

one of which $(\Psi_J \mathsf{P}_{+12})$ satisfies the Dirac equation in the Hestenes form and the other of which $(\Psi_J \mathsf{P}_{-12})$ satisfies the same equation but with the sign of the mass reversed.

As noted above, Ψ_J is generally complex whereas the Hestenes Ψ_H is real. Let Ψ_J^* be the complex conjugate of Ψ_J, obtained by replacing each scalar coefficient Ψ_J^K in the expansion (10) of Ψ_J by its complex conjugate. The property (14) can be used to replace the complex even Ψ_J in the products $\Psi_J \mathsf{P}_{\pm 12}$ by real even elements $\Psi_J^{\pm}$:

$$\Psi_J \mathsf{P}_{\pm 12} = \Psi_J^{\pm} \mathsf{P}_{\pm 12} \tag{16}$$

$$\Psi_J^{\pm} \equiv \frac{1}{2}(\Psi_J + \Psi_J^*) \pm \frac{i}{2}(\Psi_J - \Psi_J^*)\gamma_1\gamma_2 \tag{17}$$

thereby giving $\Psi_J^{\pm} \mathsf{P}_{\pm 12}$ as solutions of the Hestenes form of the Dirac equation with the correct or opposite sign on the mass term. Note that $\mathsf{P}_{\pm 12}$ is even and therefore preserves the even property of the spinor. It is also complex, but since the Hestenes equation (11) is real and linear, the real part of any complex solution is also a solution. The real part of $\Psi_J^{\pm} \mathsf{P}_{\pm 12}$ is just $\frac{1}{2}\Psi_J^{\pm}$, and consequently the two spinors $\Psi_J^{\pm}$ (17) are real even solutions of the Dirac equation in the Hestenes form (11) with the two signs of the mass term. The products $\Psi_J^{\pm}\gamma_1\gamma_2$ are similar solutions. The sum and difference of such solutions give the real and imaginary parts of the Joyce spinor Ψ_J:

$$\Psi_J = \frac{1}{2}(\Psi_J^+ + \Psi_J^-) + \frac{i}{2}(\Psi_J^+ - \Psi_J^-)\gamma_1\gamma_2. \tag{18}$$

This completes the demonstration relating every solution of the Joyce equation (7) to sums and differences of solutions to the Hestenes form of the Dirac equation with correct and reversed signs on the mass term. The close association of Hestenes and Joyce spinors is reasonable considering that the current density $\boldsymbol{J}$ has the same form, namely the vector part of $\Psi\gamma_0\tilde{\Psi}$ ($\tilde{\Psi}$ is the reversion of Ψ), if Ψ belongs to the real algebra $C\ell_{1,3}$.

It is noted in passing that similar arguments, using both pairs of commuting projectors $\mathsf{P}_{\pm 12}$ and $\mathsf{P}_{\pm 0}$, can relate a general solution Ψ of the Dirac equation (5) to four independent solutions $\Psi_H \in C\ell_{1,3}^+$ of the Hestenes equation (11) (with the correct mass term):

$$\begin{aligned} \Psi_{H1} &= \frac{1}{2}(\Psi_+ + \Psi_+^*)\gamma_1\gamma_2 + \frac{1}{2i}(\Psi_- - \Psi_-^*)\gamma_0, \qquad & \Psi_{H2} &= \Psi_{H1}\gamma_1\gamma_2 \\ \Psi_{H3} &= \frac{i}{2}(\Psi_+ - \Psi_+^*)\gamma_1\gamma_2 + \frac{1}{2}(\Psi_- + \Psi_-^*)\gamma_0, \qquad & \Psi_{H4} &= \Psi_{H3}\gamma_1\gamma_2 \end{aligned} \tag{19}$$

where Ψ_+ and Ψ_- are the even and odd parts of Ψ.

4. Symmetry of the Hestenes equation

Part of Joyce's stated reason for seeking an alternative algebraic form of the Dirac equation was that he viewed the Hestenes form (11) as giving special status to given directions in space. In particular, because equation (11) contains the $\gamma_1\gamma_2$ bivector, it was felt that the corresponding plane was singled out. On this basis it might be disappointing that every solution to the Joyce equation (7) is a combination of solutions to two equations of the Hestenes form. However, the asymmetry that Joyce saw in the Hestenes equation is only apparent, as explained below.

The principal advantages of the Hestenes formulation of the Dirac equation are (1) that it acts in the real Dirac algebra $C\ell_{1,3}$ rather than in the more traditional complex Dirac algebra used by most authors as well as by Joyce, and (2) it offers unambiguous geometrical interpretations for expressions in the theory. The fact that the spinor of the Hestenes formulation is an even element and that the Hestenes equation preserves its evenness suggests that the Dirac theory can also be formulated in the real Pauli algebra $C\ell_3$, which is isomorphic to $C\ell_{1,3}^+$. Indeed there is a very simple covariant formulation [8–10] using paravectors [11] of $C\ell_3$, and this is closely related to formulations in biquaternions and 2×2 matrices [12–14]. Further background and references can be found in a couple of recent papers [15, 16].

In both the $C\ell_3$ formulation and in Hestenes' analysis, the spinor plays the role of a relativistic transformation amplitude from the reference frame of the fermion to the laboratory frame. The orientation of the reference frame is not significant since global gauge transformations $\Psi_H \to \Psi_H R$, where R is a fixed spatial rotor, can rotate it arbitrarily. It is therefore of no physical consequence that the particular bivector $\gamma_1\gamma_2$ appears in the Hestenes equation (11) [17].

5. Plane-wave solutions

In section 10 of his paper, Joyce sought plane-wave solutions of the form $\Psi_J = A\exp(ik^\mu x_\mu) = A\exp(i\omega t - ikx)$, where A is a constant element, to his equation (7). Differentiation gives the condition[2]

$$-(\omega + \gamma_0\gamma_1 k)A = m\gamma_0 A\gamma_0. \tag{20}$$

Solving for A, substituting into the left-hand side of (20), and noting $(\gamma_0\gamma_1)^2 = 1$, one finds $\omega^2 = k^2 + m^2$ for nonvanishing A. Furthermore, A has an expansion analogous to that of Ψ_J (10) and is conveniently split into one part that commutes with γ_0 and another that anticommutes with it:

$$A = A_+ + A_- \tag{21}$$

$$\gamma_0 A_\pm \gamma_0 = \pm A_\pm \tag{22}$$

$$A_+ = a + b\gamma_1\gamma_2 + c\gamma_2\gamma_3 + d\gamma_3\gamma_1 \tag{23}$$

$$A_- = \gamma_0\gamma_1(a' + b'\gamma_1\gamma_2 + c'\gamma_2\gamma_3 + d'\gamma_3\gamma_1). \tag{24}$$

A necessary and sufficient relation for the nontrivial solution of (20) is easily seen to be

$$A_- = -\frac{\omega + m}{k}\gamma_0\gamma_1 A_+ \tag{25}$$

and the equivalent relation

$$A_+ = -\frac{\omega - m}{k}\gamma_0\gamma_1 A_-. \tag{26}$$

Except for the sign of k, the solution is equivalent to that found by Joyce, and as he points out, there are eight independent solutions for a given eigenvalue of the momentum $\boldsymbol{p} = k\boldsymbol{e}_1$ (four for each sign of $\omega = \pm\sqrt{k^2 + m^2}$), and this is twice as many as for the Dirac equation. The presence of eight such solutions reflects the fact, discussed above, that Joyce's spinor can be split into parts that solve both the usual Dirac equation and same equation with a reversed mass sign. Explicitly, if $b = ia$ and $d = ic$, the part $\Psi_J \mathsf{P}_{-12}$ of the plane-wave solution Ψ_J vanishes and Ψ_J satisfies the usual Dirac equation, whereas if $b = -ia$ and $d = -ic$, the part $\Psi_J \mathsf{P}_{+12}$ vanishes and Ψ_J satisfies the Dirac equation with mass $-m$. Otherwise, the plane wave Ψ_J is a solution of the Dirac equation only in the limit $m \to 0$.

Acknowledgments

The author is grateful for the hospitality of A Lasenby and the Cavendish Astrophysics Group during a sabbatical leave spent there. He also acknowledges helpful discussions with C Doran and the financial support of the Natural Sciences and Engineering Research Council of Canada.

[2] Joyce's related equation (22) differs in the sign of k, probably stemming from the identification of the component k or the definition of ∇ (his $\boldsymbol{d}$), equation (2). My explicit identification is $k \equiv k^1$ and $x \equiv x^1$.

References

[1] Joyce W P 2001 *J. Phys. A: Math. Gen.* **34** 1991
[2] Hestenes D 1966 *Spacetime Algebra* (New York: Gordon and Breach)
[3] Hestenes D 1975 *J. Math. Phys.* **16** 556
[4] Hestenes D 1981 Geometry of the Dirac theory *A Symposium on the Mathematics of Space-Time* ed J Keller (Mexico City: Facultad de Quimica, Universidad Nacional Autónoma de México) p 67
[5] Hestenes D 1997 Real Dirac theory *The Theory of the Electron (Advances in Applied Clifford Algebras vol 7 (S))* ed J Keller and Z Oziewicz (Cuautitlán: Facultad de Estudios Superiores Cuautitlán, Universidad Nacional Autónoma de México) p 97
[6] Baylis W E 1996 *Clifford (Geometric) Algebra with Applications to Physics, Mathematics, and Engineering* (Boston, MA: Birkhäuser)
[7] Lounesto P 1997 *Clifford Algebras and Spinors* (Cambridge: Cambridge University Press)
[8] Baylis W E 1992 *Phys. Rev.* A **45** 4293
[9] Baylis W E 1997 Eigenspinors and electron spin *The Theory of the Electron (Advances in Applied Clifford Algebras vol (S))* ed J Keller and Z Oziewicz (Cuautitlán: Facultad de Estudios Superiores Cuautitlán, Universidad Nacional Autónoma de México) p 197
[10] Daviau C 1998 Dirac equation in the space Clifford algebra *Clifford Algebras and their Applications in Mathematical Physics: Aachen 1996* ed V Dietrich, K Habetha and G Jank (Dordrecht: Kluwer) p 67
[11] Baylis W E 1999 *Electrodynamics: a Modern Geometric Approach* (Boston, MA: Birkhäuser)
[12] Lanczos C 1929 *Z. Phys.* **57** 447
Lanczos C 1929 *Z. Phys.* **57** 474
Lanczos C 1929 *Z. Phys.* **57** 484
[13] Gürsey F 1956 *Nuovo Cimento* **3** 988
[14] Rastall P 1988 *Nuovo Cimento* B **101** 479
[15] Gsponer A and Hurni J-P 2001 *Found. Phys. Lett.* **14** 77
[16] Fauser B 2001 *Int. J. Theor. Phys.* **40** 399
[17] Doran C, Lasenby A, Gull S, Somaroo S and Challinor A 1996 Spacetime algebra and electron physics *Advances in Imaging and Electron Physics* vol 95 (San Diego: Academic) p 271